

In this activity you start with the plot of a cubic function, $f(x)$. Your job is to investigate the behavior of the slope of the function—its derivative as defined by the line tangent to the function. The goal of your investigation is to be able to predict and trace the resulting derivative function (the slope of the graph as a function of the x -values).

SKETCH AND INVESTIGATE

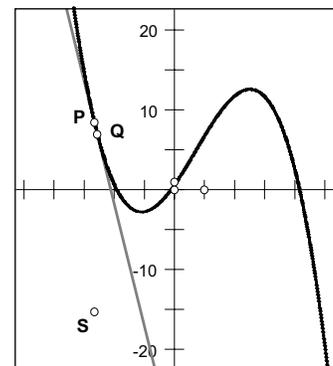
1. Open the document **PlotDerivative.gsp**. Page 1 shows the plot of a cubic function. Sliders on the left allow you to vary the function.
2. Use the **Line** tool to construct a line with both construction points on the plot. Label the left point P and the right point Q with the **Text** tool.
3. With just point P selected, measure its x -coordinate by choosing **Abscissa (x)** from the Measure menu.

Select the line and choose **Slope** from the Measure menu.

If the command is not enabled, make sure you have exactly two measurements selected, and nothing else.

4. Measure the slope of the line and move point Q relatively close to point P .
5. Select x_P and the slope measurement, $slopePQ$, in that order. Choose **Plot as (x, y)** from the Graph menu. Label this point S .

- Q1** $slopePQ$, or the y -coordinate of point S , represents an approximation of the slope or derivative of $f(x)$ at point P . Explain why it is only an approximation and how you can minimize the approximation error.



To see the plot of the slope or “derivative” function, you can track the behavior of point S as points P and Q move along the graph.

6. With point S selected, choose **Display | Trace Plotted Point**.
- Q2** Using the **Arrow** tool, select points P and Q only; then click either point and drag them slowly along the curve. What shape does the slope function trace out?
7. Erase the existing traces by pressing the Esc key (twice) or choosing **Erase Traces** from the Display menu.
8. Your trace will be smoother if you animate the points rather than drag them by hand. With both points on the graph selected, choose **Animate Points** from the Display menu. Let the points go far enough to make a complete circuit back to their starting point. Don't press the *Reverse* button.

If the shape is ragged, try again with the points closer together.

You can speed them up by pressing the Up Arrow in the Motion Controller.

A full cycle means going all the way to the right, then all the way to the left, and then back to your starting points.

- Q3** Do you notice any difference between the left-to-right trip and the right-to-left trip of the points? If so, why do you suppose this difference exists?
- Q4** Press the Pause button in the Motion Controller, and move the points farther apart. With points P and Q selected, press Pause again. Allow the motion to continue for a full cycle. What do you notice about the discrepancy between the two passes of the points?
- Q5** Pause the animation, move points P and Q as close together as you can, and erase your traces. Again select both points and release the Pause button. Allow the motion to continue for a full cycle, and then stop. What do you notice this time about the discrepancy between the two passes of the points? (Don't erase the trace this time.)

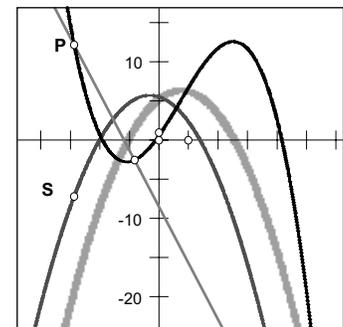
By using two arbitrary points, you have plotted point S with a y -value of $\frac{f(x_Q) - f(x_P)}{x_Q - x_P}$. By taking point Q close to point P , your y -value approximates $\lim_{Q \rightarrow P} \frac{f(x_Q) - f(x_P)}{x_Q - x_P}$, or the derivative at $x = x_P$. Now, if you fix point Q 's x -coordinate a set distance, h , from point P 's x -coordinate, you'll still be approximating the derivative. You'll just be using another form of the definition and approximating $\lim_{h \rightarrow 0} \frac{f(x_P + h) - f(x_P)}{h}$ instead.

Click a measurement or function to enter it into the calculator.

- 9. Select point Q and choose **Split Point from Function Plot** from the Edit menu. (Still, don't erase the trace.)
- 10. Choose **Calculate** from the Number menu and calculate $x_P + h$. Then calculate $f(x_P + h)$.
- 11. Plot the point $(x_P + h, f(x_P + h))$ by selecting $x_P + h$, then selecting $f(x_P + h)$, and choosing **Plot as (x, y)** from the Graph menu.
- 12. Select point Q and the new point and choose **Edit | Merge Points**.

Use the **Text** tool to relabel the merged point Q .

Now you have a secant line and plotted point similar to the ones you constructed by hand in steps 1–6, but with the value h determining the horizontal separation between the two points on the function. This allows you to animate just point P and know that the points on the function will remain separated by the same amount. It also allows you to actually construct the locus of the plotted point S .



- 13. Turn off tracing for point S (but don't erase the previous trace). With point S still selected, select point P and choose **Construct | Locus**.
- 14. Experiment with the slider for h to see how it affects the locus of point S .

- Q6** What must you do with your slider for h to make the locus match the trace you made in Q5? Explain what you are doing mathematically when you match up the trace and the locus by adjusting h . (When you are done, press the Esc key twice to erase the trace.)

EXPLORE MORE

In the activity Going Off on a Tangent, you use the slope of the tangent line to find the derivative at a point. Could you make a rough sketch of the derivative by tracking how the tangent line's slope changes?

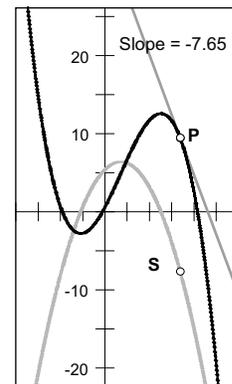
15. Select the locus and point S , and then choose **Hide/Show** from the Action Buttons submenu in the Edit menu.
16. Press the *Hide Objects* button. Then adjust any of the sliders a – d to make a new cubic function.
17. Adjust the h slider as close as possible to $h = 0$ so that you have a good approximation of the tangent line at $x = x_p$. Move point P along the function plot and observe how the slope of this approximate tangent line changes. When is the slope of the line positive? Negative? Zero?
18. Predict what you think the locus of point S will look like and make a little sketch of your prediction in the margin.
19. Press the *Show Objects* button. Compare your prediction with the locus shown. How did you do?

Check the status line or the Motion Controller to see if point P is selected. If it isn't, click the point again.

By comparing the slope of the secant line (approximated tangent line) with the location of point S as you drag point P , you can discover the relationship between them. This will help you predict what any function's derivative plot will look like. The following questions will help you discover these relationships. To answer these questions, move point P along the function and focus on the slope of the line and the location of point S .

- Q7** When the slope of the line is positive, what can you say about the location of point S ? When the slope of the line is negative? When the slope is 0?
- Q8** Drag P slowly from left to right. When the *slope* of the line is increasing, what can you say about the location of point S ? When the slope of the line is decreasing?

20. Check your answers by pressing the *Hide Objects* button and adjusting the sliders to create a different function plot.



Make sure you are looking at the slope of the line and not the function f .

Move point P along the function, make a prediction for the locus, and sketch it in the margin. Then show the locus and see how you did this time.

21. Once you are satisfied you can predict the graph for these cubic functions, double-click the expression for $f(x)$ and edit it to $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Repeat step 18.
 22. Adjust the sliders to create a variety of fourth-degree polynomial functions. (Make sure to try some negative values for the sliders.) Try to predict the locus in each case.
- Q9** Describe the graph that point S sweeps out. Is it consistent with your answers to Q7 and Q8?

Try predicting what the locus will look like for other functions (see the following examples), and then showing the locus to check your predictions.

For each function, describe the locus traced by point S for the functions you have created and check that they are consistent with your answers to Q7 and Q8.

Examples to try:

$$f_1(x) = a^x \text{ for } a > 1$$

$$f_2(x) = a^x \text{ for } 0 < a < 1$$

$$f_3(x) = a \sin(bx)$$

$$f_4(x) = \frac{a}{x^2 + b}$$

$$f_5(x) = \sqrt{9 - x^2}$$

